THE ALGEBRAS OF SEMI-INVARIANTS OF EUCLIDEAN QUIVERS

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ABSTRACT. We give a new short proof of Skowroński and Weyman's theorem about the structure of the algebras of semi-invariants of Euclidean quivers, in the case of quivers without oriented cycles and in characteristic zero. Our proof is based essentially on Derksen and Weyman's result about the generators of these algebras and properties of Schofield semi-invariants.[1]

Introduction

Let Q be a connected and finite quiver. Define $Rep(Q, \underline{\alpha})$ the affine variety of representations of Q of dimension vector $\underline{\alpha}$. We are interested in the action of the group $SL(\underline{\alpha}) := \prod_{x \in Q_0} SL(\underline{\alpha}(x))$ on this variety. In particular, we look at the algebra of semi-invariants $SI(Q, \underline{\alpha}) := K[Rep(Q, \underline{\alpha})]^{SL(\underline{\alpha})}$. By Sato-Kimura's lemma ([SK], Section 4 Proposition 5) it follows that for Dynkin quivers the algebra of semi-invariants is a polynomial algebra.

In ([SW], Theorem 1), Skowroński and Weyman prove the following theorem:

Theorem 0.1 *For each dimension vector $\underline{\alpha}$, the algebra $SI(Q, \underline{\alpha})$ is either a polynomial algebra or a quotient of a polynomial algebra by a principal ideal if and only if Q is a Dynkin quiver or a Euclidean quiver.*

In particular, they give an explicit description of $SI(Q, \underline{d})$ where Q is a Euclidean quiver (also with oriented cycles) and $\underline{d}$ a dimension vector ([SW], Theorem 21). On the other hand in ([Sh], Theorem 8.6) Shmelkin gives an independent proof of Theorem 0.1 based on some Luna's results ([Lu1], [Lu2]).

In this paper, assuming that Q is a quiver without oriented cycles and K is an algebraically closed field of characteristic zero, we find the same presentation of the algebra $SI(Q, \underline{d})$, Q a Euclidean quiver and $\underline{d}$ a dimension vector, in the sense described in [SW], using new short methods. Our proof is based on Derksen and Weyman's theorem about generators of algebras of semi-invariants for quivers without oriented cycles ([DW], Theorem 1), on some properties of Schofield semi-invariants ([DW], Lemma 1) and on Derksen, Schofield and Weyman's theorem relating the dimension of weight spaces of semi-invariants to the number of subrepresentations of a general quiver representation ([DSW], Theorem 1).

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The paper is organized as follows. In Section 1, we recall some results about Schofield semi-invariants. In Section 2, we recall some facts about Euclidean quivers. In Section 3, we formulate the results on the structure of the algebras of semi-invariants of Euclidean quivers. In Section 4, we provide the proofs of the results stated in Section 3.

1   Preliminary results

Let K be an algebraically closed field of characteristic zero.
A **quiver** is a directed graph $Q = (Q_0, Q_1)$, where Q_0 is the set of vertices and Q_1 is the set of arrows. Each arrow has its head ha and tail ta, both in Q_0;

$$ta \xrightarrow{a} ha.$$

We assume that the quiver Q is **finite** (that is, Q_0 and Q_1 are finite) and **connected**.
A **path** p is a sequence $a_1 a_2 \cdots a_m$ of arrows such that $ta_i = ha_{i+1}$ for $i = 1, \ldots, m-1$. For every vertex $x \in Q_0$ we also have a trivial path e_x with $he_x = te_x = x$. We say that Q has no oriented cycles if there is no nontrivial path p such that $tp = hp$.
We will assume from now on that Q has **no oriented cycles**.

A **representation** V of Q is a collection $(V(x) \mid x \in Q_0)$ of finite dimensional K-vector spaces together with a collection of K-linear maps $(V(a) : V(ta) \to V(ha) \mid a \in Q_1)$. For every representation V we define the **dimension vector** $\underline{d}_V : Q_0 \to \mathbb{N}$ of V by $\underline{d}_V(x) := dim_K V(x)$, $x \in Q_0$.
Suppose that V and W are representations of a quiver Q. A **morphism** $f : V \to W$ is a collection of K-linear maps $(f(x) : V(x) \to W(x) \mid x \in Q_0)$ such that for each arrow $a \in Q_1$

$$f(ha)V(a) = W(a)f(ta).$$

If moreover, $f(x)$ is invertible for each $x \in Q_0$, then f is called an **isomorphism**. We denote the linear space of morphisms from V to W by $Hom_Q(V, W)$.

Let Q be a quiver as above and $\underline{\alpha}$ a dimension vector. We can view a representation of Q of dimension vector $\underline{\alpha}$ as an element in

$$Rep\,(Q, \underline{\alpha}) := \bigoplus_{a \in Q_1} Hom(K^{\underline{\alpha}(ta)}, K^{\underline{\alpha}(ha)}).$$

The group $GL(\underline{\alpha}) := \prod_{x \in Q_0} GL(\underline{\alpha}(x))$ and its subgroup $SL(\underline{\alpha})$ act on the representation space $Rep\,(Q, \underline{\alpha})$ in the following way:

$$(\phi \cdot V)(a) := \phi(ha)V(a)\phi(ta)^{-1}$$

where $\phi = (\phi(x) \in GL(\underline{\alpha}(x)) \mid x \in Q_0) \in GL(\underline{\alpha})$ and $V \in Rep(Q, \underline{\alpha})$. We will look at the ring of $SL(Q, \underline{\alpha})$-invariants which is isomorphic to **the ring of semi-invariants**

$$SI(Q, \underline{\alpha}) := \bigoplus_{\chi \in char(GL(Q, \underline{\alpha}))} SI(Q, \underline{\alpha})_\chi$$

where

$$SI(Q, \underline{\alpha})_\chi := \{f \in K[Rep\,(Q, \underline{\alpha})] \mid g \cdot f = \chi(g)f, \; g \in GL(\underline{\alpha})\}.$$

Suppose that $\chi : GL(\underline{\alpha}) \to K^*$ is a character. Such character always looks like

$$(\phi(x) \in GL(\underline{\alpha}(x)) \mid x \in Q_0) \to \prod_{x \in Q_0} det(\phi(x))^{\sigma(x)}.$$

Here any map $\sigma : Q_0 \to \mathbb{Z}$ is called a **weight**.

Next, we recall Schofield semi-invariants [Sc], as they are the main objects we shall use in our proofs.
For representations V and W of Q there is a canonical exact sequence ([Ri1]):

$$0 \to Hom_Q(V, W) \to \bigoplus_{x \in Q_0} Hom(V(x), W(x)) \overset{d_V^W}{\to}$$

$$\overset{d_V^W}{\to} \bigoplus_{a \in Q_1} Hom(V(ta), W(ha)) \to Ext_Q(V, W) \to 0.$$

The map d_V^W is defined by

$$(\phi(x) \mid x \in Q_0) \mapsto (W(a)\phi(ta) - \phi(ha)V(a) \mid a \in Q_1).$$

For $\underline{\alpha}, \underline{\beta}$ dimension vectors, we define the **Euler form**

$$\langle \underline{\alpha}, \underline{\beta} \rangle = \sum_{x \in Q_0} \underline{\alpha}(x)\underline{\beta}(x) - \sum_{a \in Q_1} \underline{\alpha}(ta)\underline{\beta}(ha).$$

Let us choose the dimension vectors $\underline{\alpha}$ and $\underline{\beta}$ such that $\langle \underline{\alpha}, \underline{\beta} \rangle = 0$. Then for every $V \in Rep(Q, \underline{\alpha})$ and $W \in Rep(Q, \underline{\beta})$ the matrix of d_W^V is a square matrix. Following [Sc], we can define the semi-invariant $c \in K[Rep(Q, \underline{\alpha}) \times Rep(Q, \underline{\beta})]$ by $c(V, W) := det(d_W^V)$.
For a fixed V, the restriction of c to $\{V\} \times Rep(Q, \underline{\beta})$ defines a semi-invariant

$$c^V := c(V, -) \in K[Rep(Q, \underline{\beta})]^{SL(\underline{\beta})} = SI(Q, \underline{\beta}).$$

Similarly, for a fixed W, the restriction of c to $Rep(Q, \underline{\alpha}) \times \{W\}$ defines a semi-invariant

$$c_W := c(-, W) \in K[Rep(Q, \underline{\alpha})]^{SL(\underline{\alpha})} = SI(Q, \underline{\alpha}).$$

The semi-invariants c^V and c_W are called **Schofield semi-invariants** corresponding to V and W respectively.

These semi-invariants have the following important properties ([Sc] Lemma 1.4; [DW] Lemma1):

Proposition 1.1

a) The semi-invariant c^V lies in $SI(Q, \underline{\beta})_{\langle \underline{\alpha}, - \rangle}$ for every $V \in Rep(Q, \underline{\alpha})$.

b) The semi-invariant c_W lies in $SI(Q, \underline{\alpha})_{-\langle -, \underline{\beta} \rangle}$ for every $W \in Rep(Q, \underline{\beta})$.

c) Let
$$0 \to V' \to V \to V'' \to 0$$
be an exact sequence of representations, with $\underline{\alpha}' := \underline{d}_{V'}$. If $\langle \underline{\alpha}', \underline{\beta} \rangle = 0$, then $c^V = c^{V'} c^{V''}$. If $\langle \underline{\alpha}', \underline{\beta} \rangle < 0$, then $c^V = 0$.

d) Let
$$0 \to W' \to W \to W'' \to 0$$
be an exact sequence of representations, with $\underline{\beta}' := \underline{d}_{W'}$. If $\langle \underline{\alpha}, \underline{\beta}' \rangle = 0$, then $c_W = c_{W'} c_{W''}$. If $\langle \underline{\alpha}, \underline{\beta}' \rangle > 0$, then $c_W = 0$.

e) $c^V(W) = 0 \Leftrightarrow Hom_Q(V, W) \neq 0 \Leftrightarrow Ext_Q(V, W) \neq 0$. $\diamondsuit$

Let's recall two important results on semi-invariants which our proofs are based on.

Theorem 1.2 ([DW] Theorem 1, [Ch] Corollary 2.5) *Let Q be a quiver without oriented cycles, $\underline{\beta}$ a dimension vector, σ a weight.*

a) For any dimension vector $\underline{\gamma}$, a weight space $SI(Q, \underline{\gamma})_\sigma$ can be nonzero only for weights satisfying $\sigma(\underline{\gamma}) = \sum_{x \in Q_0} \sigma(x) \underline{\gamma}(x) = 0$.

b) If there is no dimension vector $\underline{\alpha}$ such that σ and $\langle \underline{\alpha}, - \rangle$ determine the same character of $GL(\underline{\beta})$, then $SI(Q, \underline{\beta})_\sigma = 0$.

c) If $\sigma = \langle \underline{\alpha}, - \rangle$ with $\langle \underline{\alpha}, \underline{\beta} \rangle = 0$, then $SI(Q, \underline{\beta})_\sigma$ is spanned as a vector space by the semi-invariants c^V for $V \in Rep(Q, \underline{\alpha})$. $\diamondsuit$

It follows from Proposition 1.1 and Theorem 1.2 that $SI(Q, \underline{\beta})$ is generated as a K-algebra by Schofield semi-invariants c^V, for V indecomposable representation of Q with $\langle \underline{d}_V, \underline{\beta} \rangle = 0$.

Before giving the next result, we recall the notion of general representation ([Ka],[Ka1] Section 4).

We say that a **general representation** with dimension vector $\underline{d}$ has a certain property, if all representations in some Zariski open (and dense) subset of the space of $\underline{d}$-dimensional representations have that property. We say that

$$\underline{d} = \underline{d}_1 + \underline{d}_2 + \ldots + \underline{d}_r$$

is the generic decomposition of a dimension vector $\underline{d}$ if a general representation W of dimension vector $\underline{d}$ has a decomposition $W = W_1 \oplus W_2 \oplus \ldots \oplus W_r$, where each W_i is indecomposable of dimension vector $\underline{d}_i$.

Theorem 1.3 ([DSW] Theorem 1) *Let Q be a quiver without oriented cycles and $\underline{\alpha}$ and $\underline{\beta}$ two dimension vectors. Let $N(\underline{\beta}, \underline{\alpha})$ be the number of $\underline{\beta}$-dimensional subrepresentations of a general $\underline{\alpha}$-dimensional representation, and $M(\underline{\beta}, \underline{\alpha})$ the dimension of the space of semi-invariants of weight $\langle \underline{\beta}, - \rangle$ on the representation space of dimension vector $\underline{\gamma} := \underline{\alpha} - \underline{\beta}$. If $\langle \underline{\beta}, \underline{\gamma} \rangle = 0$, then $N(\underline{\beta}, \underline{\alpha}) = M(\underline{\beta}, \underline{\alpha})$.* $\diamond$

2   Euclidean quivers

We recall some facts about Euclidean quivers.
Let Q be a Euclidean quiver without oriented cycles of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, i.e a quiver for which the underlying graph is one of the following type:

$$
\tilde{A}_n : \quad
\begin{array}{ccccccc}
 & c_1 & \!\!-\!\! & \cdots & \!\!-\!\! & c_u & \\
\nearrow & & & & & & \searrow \\
a & & & & & & b \\
\searrow & & & & & & \nearrow \\
 & d_1 & \!\!-\!\! & \cdots & \!\!-\!\! & d_v &
\end{array}
$$

where $u + v = n - 1$,

$$
\tilde{D}_n : \quad
\begin{array}{ccccccc}
a_1 & & & & & & b_1 \\
 & \searrow & & & & \nearrow & \\
 & & z_1 & \!\!-\!\! \cdots \!\!-\!\! & z_{n-3} & & \\
 & \nearrow & & & & \searrow & \\
a_2 & & & & & & b_2
\end{array}
$$

$$
\tilde{E}_6 : \quad
\begin{array}{ccccc}
 & & c_1 & & \\
 & & | & & \\
 & & c_2 & & \\
 & & | & & \\
a_1 & \!\!-\!\! \ a_2 \ \!\!-\!\! & z & \!\!-\!\! \ b_2 \ \!\!-\!\! & b_1
\end{array}
$$

$$
\tilde{E}_7 : \quad
\begin{array}{ccccccc}
 & & & c & & & \\
 & & & | & & & \\
b_1 & \!\!-\!\! \ b_2 \ \!\!-\!\! & b_3 & \!\!-\!\! \ z \ \!\!-\!\! & a_3 & \!\!-\!\! \ a_2 \ \!\!-\!\! & a_1
\end{array}
$$

$$
\tilde{E}_8 : \quad
\begin{array}{ccccccc}
 & & c & & & & \\
 & & | & & & & \\
b_1 & \!\!-\!\! \ b_2 \ \!\!-\!\! & z & \!\!-\!\! \ a_5 \ \!\!-\!\! & a_4 & \!\!-\!\! \ a_3 \ \!\!-\!\! \ a_2 \ \!\!-\!\! & a_1
\end{array}
$$

By [DR] Proposition 1.2, the quadratic form $q_Q : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ defined by

$$q_Q(\underline{\alpha}) := \sum_{v \in Q_0} \underline{\alpha}(v)^2 - \sum_{a \in Q_1} \underline{\alpha}(ta)\underline{\alpha}(ha)$$

is positive semi-definite and there exists a unique dimension vector $\underline{h} \in \mathbb{N}^{Q_0}$ such that $\mathbb{Z}\underline{h}$ is the radical of q_Q ([DR] page 9).

Following [DR] Section 1, we define the **defect** of a module V as $\partial(\underline{d}_V) := \langle \underline{h}, \underline{d}_V \rangle$, we say that an indecomposable representation V is *preprojective, regular* or *preinjective* if and only if the defect of V is negative, zero or positive, respectively. As we shall see later (Section 4.1), we are interested in regular modules, so we give more details about them.

Regular representations of Q form an abelian category $Reg_K(Q)$ ([DR] Proposition 3.2), so we may speak of regular composition series, simple regular objects, etc., referring to composition series, simple objects, etc. inside the category $Reg_K(Q)$. The category $Reg_K(Q)$ is serial ([DR] Theorem 3.5): any indecomposable regular representation has a unique regular composition series, thus it is uniquely determined by its regular socle (simple regular) and by its regular length.

By [DR] Theorem 3.5, the category $Reg_K(Q)$ decomposes into a direct sum of categories R_t, with $t \in K \cup \{\infty\} = \mathbb{P}_1(K)$. We call each category R_t a **tube**. In order to describe such tubes, we need the following definition:

Definition 2.1 *Let Q be a quiver without oriented cycles. We may assume that $Q_0 = \{1, 2, \ldots, n\}$ and for every $a \in Q_1$ we have $ta < ha$. We define the* **Coxeter element**

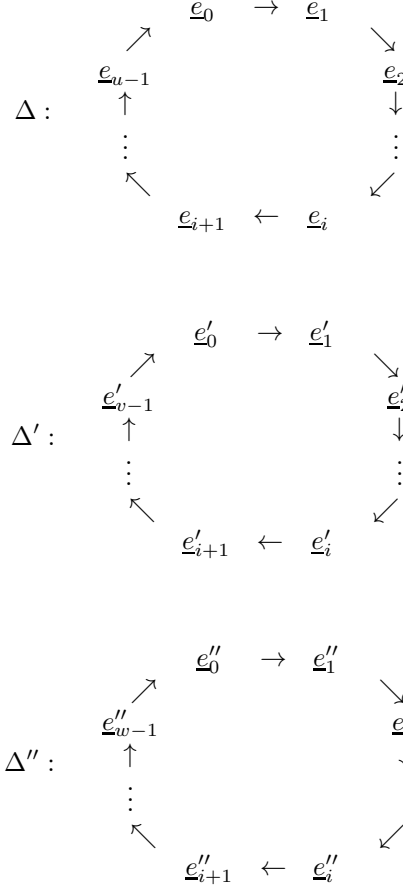$$C := \sigma_1 \sigma_2 \ldots \sigma_n$$

where each σ_i acts on dimension vectors as follows:

$$\sigma_i(\underline{\alpha})(j) = \begin{cases} \underline{\alpha}(j) & \text{if } j \neq i; \\ \sum_{a \in Q_1; ta=i} \underline{\alpha}(ha) + \sum_{a \in Q_1; ha=i} \underline{\alpha}(ta) - \underline{\alpha}(i), & \text{otherwise.} \end{cases}$$

By [DR] Lemma 1.3 and Lemma 3.3, for each simple regular representation V, the orbit of the dimension vector of V under C is always finite. In particular, a simple regular representation V with the dimension vector which is fixed by C is called **homogeneous**. In such a case, we have that the dimension vector of V is equal to $\underline{h}$. About the C-orbits of simple non homogeneous modules, we recall the following description ([DR], Section 6):

Proposition 2.2 *Let Q be a Euclidean quiver. Then there are at most three C-orbits $\Delta = \{\underline{e}_i, i \in I\}$, $\Delta' = \{\underline{e}'_i, i \in I'\}$, $\Delta'' = \{\underline{e}''_i, i \in I''\}$, of dimension vectors of non homogeneous simple regular representations of Q (I, I', I'' could be empty). We can assume $I = \{0, 1, \ldots, u-1\}$, $I' = \{0, 1, \ldots, v-1\}$, $I'' = \{0, 1, \ldots, w-1\}$ and $C(\underline{e}_i) = \underline{e}_{i+1}$ for $i \in I$ ($\underline{e}_u = \underline{e}_0$), $C(\underline{e}'_i) = \underline{e}'_{i+1}$ for $i \in I'$ ($\underline{e}'_v = \underline{e}'_0$), $C(\underline{e}''_i) = \underline{e}''_{i+1}$ for $i \in I''$ ($\underline{e}''_w = \underline{e}''_0$). $\diamond$*

Graphically we may represent them as the following polygons:

$$
\Delta: \qquad
\begin{array}{ccc}
\underline{e}_0 & \to & \underline{e}_1 \\
\nearrow & & \searrow \\
\underline{e}_{u-1} & & \underline{e}_2 \\
\uparrow & & \downarrow \\
\vdots & & \vdots \\
\nwarrow & & \swarrow \\
\underline{e}_{i+1} & \leftarrow & \underline{e}_i
\end{array}
$$

$$
\Delta': \qquad
\begin{array}{ccc}
\underline{e}'_0 & \to & \underline{e}'_1 \\
\nearrow & & \searrow \\
\underline{e}'_{v-1} & & \underline{e}'_2 \\
\uparrow & & \downarrow \\
\vdots & & \vdots \\
\nwarrow & & \swarrow \\
\underline{e}'_{i+1} & \leftarrow & \underline{e}'_i
\end{array}
$$

$$
\Delta'': \qquad
\begin{array}{ccc}
\underline{e}''_0 & \to & \underline{e}''_1 \\
\nearrow & & \searrow \\
\underline{e}''_{w-1} & & \underline{e}''_2 \\
\uparrow & & \downarrow \\
\vdots & & \vdots \\
\nwarrow & & \swarrow \\
\underline{e}''_{i+1} & \leftarrow & \underline{e}''_i
\end{array}
$$

Given a C-orbit of a simple regular module, the corresponding tube consists of the indecomposable regular modules whose regular composition factors belong to this orbit. We call the tube corresponding to the orbit of a homogeneous module a homogeneous tube.

In particular, in the sections that follow, we will need the following fact ([DR] Theorem 5.1):

Lemma 2.3 *There exists a regular map $V : K^2 \setminus \{(0,0)\} \to Rep(Q, \underline{h})$ with the following properties:*

i) $V(\varphi, \psi)$ is an indecomposable object in $R_{(\varphi : \psi)}$ for each $(\varphi, \psi) \in K^2 \setminus \{(0,0)\}$ (it has to be a simple object in $R_{(\varphi : \psi)}$ if $R_{(\varphi : \psi)}$ is a homogeneous tube).

ii) If $(\varphi : \psi) = (\gamma : \delta)$, then $V(\varphi, \psi)$ and $V(\gamma, \delta)$ are isomorphic. ◇

Denote by D_r the set of dimension vectors of all regular representations of Q. Each element $\underline{d} \in D_r$ can be written in the form

$$\underline{d} = p\underline{h} + \sum_{i \in I} p_i \underline{e}_i + \sum_{i \in I'} p'_i \underline{e}'_i + \sum_{i \in I''} p''_i \underline{e}''_i \tag{1}$$

for nonnegative integers p, p_i, p'_i, p''_i with at least one coefficient in each family $(p_i \mid i \in I), (p'_i \mid i \in I'), (p''_i \mid i \in I'')$ being zero. The decomposition in (1) is called **canonical decomposition** of $\underline{d} \in D_r$ ([Ri], Section 1). Since the only linear relations among the dimension vectors $\underline{h}, \underline{e}_i, \underline{e}'_i, \underline{e}''_i$ are the following

$$\underline{h} = \sum_{i \in I} \underline{e}_i = \sum_{i \in I'} \underline{e}'_i = \sum_{i \in I''} \underline{e}''_i,$$

we have that such decomposition is unique. For simplicity, we set

$$\underline{d}' = \sum_I p_i \underline{e}_i + \sum_{I'} p'_i \underline{e}'_i + \sum_{I''} p''_i \underline{e}''_i. \tag{2}$$

Finally, we recall the description of the generic decomposition of a regular dimension vector presented in [SW], Section 5.

We write $\sum_I p_i \underline{e}_i = \sum_{j=1}^{max(p_i)} \sum_{k=1}^{N_j(p)} \underline{v}_{j,k}$, where the dimension vectors are defined by reverse induction on $max(p_i)$ as follows. Let $s = max(p_i)$. Look at the set $M_s = \{i \in I; p_i = s\}$ and decompose it $M_s = M_{s,1} \cup \cdots \cup M_{s,t}$ into a sum of connected components (a subset of I is called connected if it is an arc of the polygon Δ). Take $t = N_s(p)$ and $\underline{v}_{s,k} = \sum_{i \in M_{s,k}} \underline{e}_i$. Then repeat the procedure for $\sum_I \tilde{p}_i \underline{\tilde{e}}_i = \sum_I p_i \underline{e}_i - \sum_{k=1}^{N_s(p)} \underline{v}_{s,k}$, because we have $max(\tilde{p}_i) = s - 1$. Similarly, we write $\sum'_I p'_i \underline{e}'_i = \sum_{j=1}^{max(p'_i)} \sum_{k=1}^{N_j(p')} \underline{v}'_{j,k}$, and $\sum''_I p''_i \underline{e}''_i = \sum_{j=1}^{max(p''_i)} \sum_{k=1}^{N_j(p'')} \underline{v}''_{j,k}$. In general, we have that the following decomposition:

$$\underline{d} = \sum_{i=1}^{p} \underline{h} + \sum_{j=1}^{max(p_i)} \sum_{k=1}^{N_j(p)} \underline{v}_{j,k} + \sum_{j=1}^{max(p'_i)} \sum_{k=1}^{N_j(p')} \underline{v}'_{j,k} + \sum_{j=1}^{max(p''_i)} \sum_{k=1}^{N_j(p'')} \underline{v}''_{j,k} \tag{3}$$

is the generic decomposition of $\underline{d}$ ([SW], Proposition 44).

3   Main results about algebras of semi-invariants

Before giving the main theorem, we recall the following notations that appear in [SW], Section 2. Let Q be a Euclidean quiver, $\underline{d} \in D_r$ with the canonical decomposition as in (1) with $p \geq 1$.

We label the vertices $\underline{e}_i, \underline{e}'_i, \underline{e}''_i$ of the polygons $\Delta, \Delta', \Delta''$ in Proposition 2.2, with the coefficients p_i, p'_i, p''_i. Now, in these new polygons, that we call $\Delta(\underline{d})$, $\Delta'(\underline{d})$, $\Delta''(\underline{d})$, we label the edge from p_k to p_{k+1} with $\underline{e}_k$, for $k = 0, \ldots, u - 2$, and the edge from p_{u-1} to p_0 with $\underline{e}_{u-1}$ (similarly for the other polygons).

We say that the labeled arc $p_i - - \cdots - -p_j$ (in clockwise orientation) of $\Delta(\underline{d})$ is **admissible** if $p_i = p_j$ and $p_i < p_k$ for all its interior labels p_k. We denote such arc by $[i, j]$. Similarly, we define admissible arcs for the polygons $\Delta'(\underline{d})$

and $\Delta''(\underline{d})$. Denote by $\mathcal{A}(\underline{d}), \mathcal{A}'(\underline{d}), \mathcal{A}''(\underline{d})$ the sets of all admissible labeled arcs in the polygons $\Delta(\underline{d}), \Delta'(\underline{d}), \Delta''(\underline{d})$.

On the other hand, we denote by $E_{i,j}$, for $i, j \in \{0, \ldots, u-1\}$ with $j \neq i+1$ and $j \neq 0$ if $i = u-1$, the indecomposable regular module of dimension vector $\underline{d}_{E_{i,j}}$ with the canonical decomposition as in (1) with $p = 0$, with socle $E_{i,i}$ and top $E_{j,j}$, where $E_{k,k}$, or simply E_k is the non homogeneous simple regular module which has dimension vector $\underline{e}_k$, as in Proposition 2.2. One gives analogous definitions for $E'_{r,s}$ and $E''_{t,m}$. Using the above notations, the dimension vector of $E_{i,j}$ is equal to the sum of all $\underline{e}_k$ which appear as edges in the polygon $\Delta(\underline{d})$ between the vertices p_j and p_{i+1} (in clockwise orientation), where we put $p_u := p_0$ (similarly for $E'_{r,s}$ and $E''_{t,m}$).

Now we are ready to state the following theorem:

Theorem 3.1 *Let Q be a Euclidean quiver, $\underline{d} \in D_r$. Let $\underline{d} = p\underline{h} + \underline{d}'$ be the canonical decomposition of $\underline{d}$, as in (1), with $p \geq 1$.*

Then the algebra $SI(Q, \underline{d})$ is generated as a K-algebra by Schofield semi-invariants corresponding to the indecomposable regular modules $E_{i,j}$, $E'_{r,s}$ and $E''_{t,m}$ for each pair $(i, j), (r, s)$ and (t, m) such that $[j, i+1] \in \mathcal{A}(\underline{d})$, $[s, r+1] \in \mathcal{A}'(\underline{d})$ and $[m, t+1] \in \mathcal{A}''(\underline{d})$ and by semi-invariants $c_0, c_1, \ldots, c_p$ of weight ∂. The ideal of relations among generators is generated by the following relations:

$$c_0 = \prod c^{E_{i,j}}, \ c_p = \prod c^{E'_{r,s}}, \ c_0 + \ldots + c_p = \prod c^{E''_{t,m}}$$

where the products are over the pairs of indices $(i, j), (r, s), (t, m)$ such that $\sum \underline{d}_{E_{i,j}} = \sum \underline{d}_{E'_{r,s}} = \sum \underline{d}_{E''_{t,m}} = \underline{h}$, respectively.

Remark: Our assumption about the dimension vector, that is $\underline{d} \in D_r$ and $\underline{d} \neq \underline{d}'$, is natural, because in the other cases $SI(Q, \underline{d})$ is a polynomial algebra. This is due to the following two facts ([SK] Section 4 Proposition 5; [Ri] Corollary 2.4, Theorem 3.2, Theorem 3.5)

Lemma 3.2 (Sato-Kimura) *Suppose that $GL(\underline{\beta})$ has a dense orbit in $Rep(Q, \underline{\beta})$. Let S be the set of all σ such that there exists an $f_\sigma \in SI(Q, \underline{\beta})_\sigma$ which is nonzero and irreducible. Then:*

a) For every weight $\sigma \in S$, we have $\dim SI(Q, \underline{\beta})_\sigma \leq 1$.

b) All weights in S are linearly independent over $\mathbb{Q}$.

c) $SI(Q, \underline{\beta})$ is the polynomial ring generated by all $f_\sigma, \sigma \in S$. $\diamond$

Proposition 3.3 *For a dimension vector $\underline{d} \in \mathbb{N}^{Q_0}$, the variety $Rep(Q, \underline{d})$ has no open $GL(\underline{d})$-orbit if and only $\underline{d} \in D_r$ and for the canonical decomposition $\underline{d} = p\underline{h} + \underline{d}'$ we have $p \geq 1$ (equivalently $\underline{d} \neq \underline{d}'$).* $\diamond$

4 Proof of Theorem 3.1

We divide the proof of Theorem 3.1 into two principal steps:

Step 1: Description of generators

By Theorem 1.2, $SI(Q, \underline{d})$ is generated, as a K-algebra, by Schofield semi-invariants $\{c^V\}$ for all V indecomposable representations of Q such that $\langle \underline{d}_V, \underline{d} \rangle = 0$. Equivalently, by the definition of the generic decomposition of a dimension vector, we consider as generators $\{c^V\}$, for all V indecomposable modules such that $\underline{d}_V$ is orthogonal to each summand of the generic decomposition of $\underline{d}$. Since vector $\underline{h}$ always appears among these summands, we'll have that all V belong to the set of regular indecomposable modules. Finally, by Proposition 1.1 e), we have

$$c^V \neq 0 \iff Hom_Q(V, W_i) = 0$$

for all W_i summands of generic decomposition of $\underline{d}$, as in (3).

Moreover, by the definition of the generic decomposition of a regular dimension vector $\underline{d}$ as in (3), Proposition 1.1 c) and Lemma 3.2, we are reduced to consider as generators of $SI(Q, \underline{d})$ the following families: an infinite family given by $\{c^{V(\varphi, \psi)}\}$, where $(\varphi, \psi) \in K^2 \setminus \{(0,0)\}$ and three finite families given by $\{c^{E_{i,j}}\}$, $\{c^{E'_{r,s}}\}$ and $\{c^{E''_{t,m}}\}$, where $E_{i,j}$, $E'_{r,s}$ and $E''_{t,m}$ are as in Section 3 and they satisfy two conditions

1. $\langle \underline{d}_{E_{i,j}}, \underline{d} \rangle = 0$, (similarly for $\underline{d}_{E'_{r,s}}$ and $\underline{d}_{E''_{t,m}}$);

2. $Hom_Q(E_{i,j}, W_k) = 0$ for all W_k summands of the generic decomposition of $\underline{d}$, (similarly for $E'_{r,s}$ and $E''_{t,m}$).

Now, remembering the notation in Section 3, we want to prove that the above conditions 1 and 2 are equivalent to the condition of admissibility of the arc $[j, i+1]$ in the polygon $\Delta(\underline{d})$ ($[s, r+1] \in \Delta'(\underline{d})$ and $[m, t+1] \in \Delta''(\underline{d})$). We start proving the following lemma

Lemma 4.1 *In the above notation, the condition 1, that is $\langle \underline{d}_{E_{i,j}}, \underline{d} \rangle = 0$, is equivalent to consider the arc $[j, i+1]$ in the polygon $\Delta(\underline{d})$ with extremes p_j and p_{i+1} that are the same number.*

Proof: As in Section 3, we write $\underline{d}_{E_{i,j}}$ as the sum of all $\underline{e}_k$ which label the edges in the arc $[j, i+1]$. Recall the following result ([DR], Lemma 3.3):

Lemma 4.2 *Let $\underline{e}_i, \underline{e}_j$ ($\underline{e}'_k, \underline{e}'_r$ and $\underline{e}''_m, \underline{e}''_n$ respectively) two dimension vectors as in Proposition 2.2, then*

$$\langle \underline{e}_i, \underline{e}_j \rangle = \begin{cases} 1 & \text{if } i = j; \\ -1 & \text{if } i = j - 1; \\ 0 & \text{otherwise}; \end{cases}$$

where $\underline{e}_{-1} := \underline{e}_{u-1}$ (we have analogous result for the other pairs). Otherwise, the value of Euler form between dimension vectors belonging to different polygons of Proposition 2.2, is zero. ◇

We put $\underline{d}_{E_{i,j}} := \sum_{k=0}^{u-1} \delta_k \underline{e}_k$, where δ_k is equal to 1, for each k such that $\underline{e}_k$ is an edge in $[j, i+1]$, and zero otherwise. Using Lemma 4.2, we have the following equalities:

$$\langle \underline{d}_{E_{i,j}}, \underline{d} \rangle = \langle \sum_{k=0}^{u-1} \delta_k \underline{e}_k, p\underline{h} + \sum_{i \in I} p_i \underline{e}_i + \sum_{j \in I'} p'_j \underline{e}'_j + \sum_{k \in I''} p''_k \underline{e}''_k \rangle =$$

$$= \langle \sum_{k=0}^{u-1} \delta_k e_k, \sum_i p_i \underline{e}_i + \sum_j p'_j \underline{e}'_j + \sum_k p''_k \underline{e}''_k \rangle =$$

$$= \sum_k \delta_k p_k \langle \underline{e}_k, \underline{e}_k \rangle + \sum_{k \neq l} \langle \delta_k \underline{e}_k, p_l \underline{e}_l \rangle = \sum_k \delta_k p_k + \sum_{k \neq l} \delta_k p_l \langle \underline{e}_k, \underline{e}_l \rangle =$$

$$= \sum_k \delta_k p_k + \sum_{l=0}^{u-1} \delta_{l-1} p_l \langle \underline{e}_{l-1}, \underline{e}_l \rangle = \sum_k \delta_k p_k - \sum_{l=0}^{u-1} \delta_{l-1} p_l =$$

$$= p_j - p_{i+1}.$$

In conclusion, we obtain

$$\langle \underline{d}_{E_{i,j}}, \underline{d} \rangle = 0 \iff p_j = p_{i+1} \tag{4}$$

as requested. $\quad \diamond$

Finally, we prove that the conditions 1 and 2 together are equivalent to the condition of admissibility of the arc $[j, i+1] \in \mathcal{A}(\underline{d})$. From now on, we'll use the short notation, $n \in [i, j]$, to say that n is such that $\underline{e}_n$ is an edge in the oriented arc $[i, j]$ of $\Delta(\underline{d})$.
By the definition of admissibility and Lemma 4.1, we are reduced to prove the following statement:

Lemma 4.3 *Let $p_j, p_{i+1} \in \Delta(\underline{d})$ such that $p_j = p_{i+1}$. Then there exists $l \in [j+1, i+1]$ such that $p_l < p_j$ if and only if there exists $\tilde{W}$ summand of the generic decomposition of $\underline{d}$ such that $Hom_Q(E_{i,j}, \tilde{W}) \neq 0$.*

Before starting the proof of Lemma 4.3, we recall a simple fact that follows by the structure of Auslander-Reiten quiver of extended Dynkin quivers ([SS], Chapter XIII.2 Theorem 2.1):

Lemma 4.4 *Let $E_{k,r}$ and $E_{m,n}$ be two regular indecomposable non homogeneous modules of Q as in Section 3. Then $Hom_Q(E_{k,r}, E_{m,n}) \neq 0$ if and only if $E_{m,n}$ with $m \in [r, k+1]$ and $n \in [m+2, r+1]$.* $\quad \diamond$

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Proof of Lemma 4.3: Let p_l be the minimal number in the arc $[j, i+1] \in \Delta(\underline{d})$ that is nearest to p_j. Graphically we have
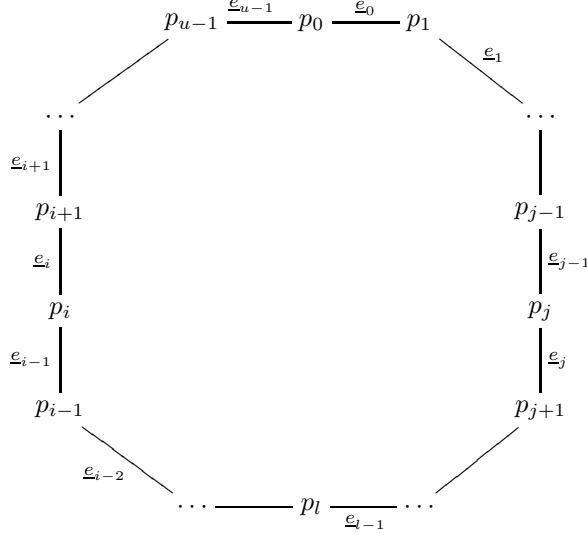
By definition of the generic decomposition of $\underline{d}$ (see (3)), there exists a summand of the generic decomposition of $\underline{d}$, $\tilde{W}$, which is isomorphic to $E_{l-1,n}$, with $n \in [l+1, j+1]$. By Lemma 4.4, we have $Hom_Q(E_{i,j}, \tilde{W}) \neq 0$, as requested.

Viceversa, if we have a summand of the generic decomposition of $\underline{d}$, $\tilde{W}$, such that $Hom_Q(E_{i,j}, \tilde{W}) \neq 0$, then, by Lemma 4.4, it is isomorphic to $E_{m,n}$, with $m \in [j, i+1]$ and $n \in [m+2, j+1]$. Now, by the hypothesis that $p_{i+1} = p_j$, no summand of the generic decomposition of $\underline{d}$ can be isomorphic to $E_{i,r}$ with $r \in [i+2, j+1]$. In all other cases, by the construction of the generic decomposition as in (3), it follows that $p_{m+1} < p_j$ and $m+1 \in [j+1, i+1]$, as requested. $\diamondsuit$

By Lemma 3.2, Proposition 3.3 and the reciprocity formula ([DW] Corollary 1), we have that

$$dim \ SI(Q, \underline{d})_{\langle \underline{d}_{E_{i,j}}, - \rangle} = dim \ SI(Q, \underline{d})_{\langle \underline{d}_{E'_{r,s}}, - \rangle} = dim \ SI(Q, \underline{d})_{\langle \underline{d}_{E''_{t,m}}, - \rangle} = 1$$

for each pair $(i, j), (r, s)$ and (t, m) such that $[j, i+1] \in \mathcal{A}(\underline{d})$, $[s, r+1] \in \mathcal{A}'(\underline{d})$ and $[m, t+1] \in \mathcal{A}''(\underline{d}))$, so each semi-invariant $c^{E_{i,j}}$, $c^{E'_{r,s}}$ and $c^{E''_{t,m}}$, where $(i, j), (r, s)$ and (t, m) are as above, spans the corresponding weight space. All the other generators, $c^{V(\varphi, \psi)}$ with $(\varphi, \psi) \in K^2 \setminus \{(0, 0)\}$ are in $SI(Q, \underline{d})_\partial$. By the next result, it follows that $dim \ SI(Q, \underline{d})_\partial = p + 1$.

Proposition 4.5 *If $\underline{d} = p\underline{h} + \underline{d}'$ is the canonical decomposition of $\underline{d}$ as in (1), then $dim \ SI(Q, \underline{d})_{m\partial} = \binom{p+m}{m}$.*

Proof: Applying the reciprocity formula, we have

$$dim\ SI(Q, \underline{d})_{m\partial} = dim\ SI(Q, m\underline{h})_{-\langle-,\underline{d}\rangle}.$$

Let c_W be an element of $SI(Q, m\underline{h})_{-\langle-,\underline{d}\rangle}$, where W is a representation of Q of dimension vector $\underline{d}$. Using the generic decomposition of $\underline{d}$ as in (3) and Proposition 1.1 d), $c_W = c_{W'}c_{W''}$ where W' and W'' are representations of dimension vectors $p\underline{h}$ and $\underline{d}'$, respectively. By Lemma 3.2 and Proposition 3.3, it follows that $dim\ SI(Q, m\underline{h})_{-\langle-,\underline{d}'\rangle} = 1$, so we have an isomorphism between $SI(Q, m\underline{h})_{-\langle-,\underline{d}\rangle}$ and $SI(Q, m\underline{h})_{-\langle-,p\underline{h}\rangle}$ given by multiplication by a non-zero semi-invariant of weight $-\langle-,\underline{d}'\rangle$.

We need to show

$$dim\ SI(Q, m\underline{h})_{-\langle-,p\underline{h}\rangle} = dim\ SI(Q, p\underline{h})_{\langle m\underline{h},-\rangle} = \binom{p+m}{m}.$$

Applying Theorem 1.3, we are reduced to count all subrepresentations of dimension vector $m\underline{h}$ in a general representation of dimension vector $(p+m)\underline{h}$. Since a general representation of dimension vector $(p+m)\underline{h}$ is a direct sum of $(p+m)$ different indecomposables of dimension vector $\underline{h}$, then it has $\binom{p+m}{m}$ subrepresentations of dimension vector $m\underline{h}$. $\quad\diamond$

Step 2: *Description of relations*

We conclude the proof of Theorem 3.1 giving a description of the relations among the generators of $SI(Q, \underline{d})$ presented in Step 1. In particular, in the proof of the following proposition, we show that the set of semi-invariants $\{c_0, \ldots, c_p\}$ that appears in the statement of Theorem 3.1, is a basis of $SI(Q, \underline{d})_\partial$. We recall the following fact of linear algebra that we use in the proof of Proposition 4.7:

Lemma 4.6 *A **Vandermonde** matrix is a $m \times n$ K-matrix, $M = (m_{i,j})$ where*

$$m_{i,j} = (\alpha_j)^{i-1},$$

for all indices i and j. If $m = n$, the determinant of a square Vandermonde matrix can be expressed as:

$$det(M) = \prod_{1 \leq i < j \leq n} (\alpha_j - \alpha_i).$$

Thus, a square Vandermonde matrix is invertible if and only if the α_i are distinct.

Proposition 4.7

(a) We have the following relations among the generators of the algebra $SI(Q, \underline{d})$

$$c_0 = \prod c^{E_{i,j}}, \qquad c_p = \prod c^{E'_{r,s}}, \qquad c_0 + \ldots + c_p = \prod c^{E''_{t,m}}, \tag{5}$$

where the products are over the pairs of indices $(i,j), (r,s), (t,m)$ such that $\sum \underline{d}_{E_{i,j}} = \sum \underline{d}_{E'_{r,s}} = \sum \underline{d}_{E''_{t,m}} = \underline{h}$, respectively.

(b) The relations in (5) are enough to generate the ideal of relations.

Proof: (a) By Step 1, we have that Schofield semi-invariants $c^{V(\varphi,\psi)}$, with $(\varphi, \psi) \in K^2 \setminus \{(0,0)\}$, span $SI(Q, \underline{d})_\partial$. Let $\underline{d} = p\underline{h} + \underline{d}'$ be the canonical decomposition of $\underline{d}$ as in (1). Using the same arguments of the proof of Proposition 4.5, we have that $c^{V(\varphi,\psi)} \in SI(Q, \underline{d})_\partial$ is equal to the product of the semi-invariant $\tilde{c}^{V(\varphi,\psi)} := c^{V(\varphi,\psi)} \in SI(Q, p\underline{h})_\partial$ and a non-zero semi-invariant $f' \in SI(Q, \underline{d}')_\partial$. We want to focus on the first factor $\tilde{c}^{V(\varphi,\psi)}$.

For a fixed $V(\varphi, \psi) \in Rep(Q, \underline{h})$, $\tilde{c}^{V(\varphi,\psi)}$ is the restriction of the function $\tilde{c} := c \in K[Rep(Q, \underline{h})] \otimes K[Rep(Q, p\underline{h})]$ to $\{V(\varphi, \psi)\} \times Rep(Q, p\underline{h})$.

On the other hand, by definition of the generic decomposition in (3), a general representation of dimension vector $p\underline{h}$ is a direct sum of p pairwise non isomorphic representations $V(\gamma_i, \delta_i)$, $i = 1, \ldots, p$ of dimension vectors $\underline{h}$.

Thus, if we define the following map

$$K^2 \times (K^2)^p \overset{\pi}{\to} Rep(Q, \underline{h}) \times Rep(Q, p\underline{h}) \qquad (6)$$

$$(\varphi, \psi, \gamma_1, \delta_1, \ldots, \gamma_p, \delta_p) \mapsto V(\varphi, \psi) \oplus \bigoplus_{i=1}^{p} V(\gamma_i, \delta_i),$$

we want to study the image in $K[\varphi, \psi] \otimes K[\gamma_i, \delta_i \mid i = 1, \ldots, p]$ of the space $SI(Q, p\underline{h})_\partial = Span_K\{\tilde{c}^{V(\varphi,\psi)} \mid (\varphi, \psi) \in K^2 \setminus \{(0,0)\}\}$ under the map π^* induced by (6).

First of all, if W is a general representation of dimension vector $p\underline{h}$, applying Proposition 1.1 d), we have that:

$$\tilde{c}^{V(\varphi,\psi)}(W) = \tilde{c}_W(V(\varphi, \psi)) = \prod_{i=1}^{p} \hat{c}_{V(\gamma_i,\delta_i)}(V(\varphi, \psi)) = \prod_{i=1}^{p} \hat{c}^{V(\varphi,\psi)}(V(\gamma_i, \delta_i)),$$

where $\hat{c} := c \in K[Rep(Q, \underline{h})] \otimes K[Rep(Q, \underline{h})]$. Thus, we are reduced to study the case $p = 1$. We need to prove the following fact:

Lemma 4.8 *Let $\hat{c}^{V(\varphi,\psi)} \in SI(Q, \underline{h})_\partial$. Then:*

i) For each (φ, ψ) and $(\gamma, \delta) \in K^2 \setminus \{(0,0)\}$, we have that

$$\hat{c}^{V(\varphi,\psi)}(V(\gamma, \delta)) = 0 \Longleftrightarrow (\varphi : \psi) = (\gamma : \delta).$$

ii) The image of $\hat{c}$ in $K[\varphi, \psi] \otimes K[\gamma, \delta]$ under the map π^* induced by (6) is a polynomial of the form $\varphi\delta - \psi\gamma$.

Proof: i) follows by Proposition 1.1 e) and the following consequence of [DR], Theorem 3.5:

$$Hom_Q(V(\varphi, \psi), V(\gamma, \delta)) \neq 0 \Longleftrightarrow (\varphi : \psi) = (\gamma : \delta).$$

ii) First of all, by *i)* follows that the image of $\hat{c}$ in $K[\varphi, \psi] \otimes K[\gamma, \delta]$ is a polynomial of the form

$$(\varphi\delta - \psi\gamma)^m = \sum_{j=0}^{m} \binom{m}{j} (\varphi\delta)^{m-j} (-\psi\gamma)^j,$$

$m \geq 1$, where the pairs of variables (φ, ψ) and (γ, δ) are two generators of a homogeneous coordinate ring of $\mathbb{P}_1(K)$, so they are linearly independent, respectively. We want to prove that $m = 1$.

Suppose $m > 1$. Then, the dimension of the image in $K[\varphi, \psi] \otimes K[\gamma, \delta]$ of the vector space $SI(Q, \underline{h})_\partial = Span_K\{\hat{c}^{V(\varphi,\psi)} \mid (\varphi, \psi) \in K^2 \setminus \{(0,0)\}\}$ would be bigger than 2, a contradiction since by Proposition 4.5 we have that $dim_K SI(Q, \underline{h})_\partial = 2$. Indeed, for example, the image of $Span_K\{\hat{c}^{V(\varphi,\psi)} \mid (\varphi, \psi) \in K^2 \setminus \{(0,0)\}\}$ contains the vector space

$$P := Span_K\{\delta^m, \gamma^m, \sum_{j=0}^{m} \binom{m}{j} (\delta)^{m-j} (-\gamma)^j\},$$

which has dimension equal to 3. $\diamondsuit$

Now, if $p > 1$, by Lemma 4.7 *ii)*, we have that the image of $\tilde{c}$ is equal to the polynomial

$$f := \prod_{i=1}^{p} (\varphi\delta_i - \psi\gamma_i).$$

Without losing generality, we put $\delta_i = 1$ for all i (similarly if we put $\gamma_i = 1$). Then, we have that

$$f = a_0(\gamma_1, \ldots, \gamma_p)\varphi^p + a_1(\gamma_1, \ldots, \gamma_p)\varphi^{p-1}\psi + \ldots + a_p(\gamma_1, \ldots, \gamma_p)\psi^p,$$

where $a_i(\gamma_1, \ldots, \gamma_p)$ are the elementary symmetric functions in $\gamma_1, \ldots, \gamma_p$.

In particular, the image of $SI(Q, p\underline{h})_\partial$ is equal to $Span_K\{f_{(\overline{\varphi}, \overline{\psi})}, (\overline{\varphi}, \overline{\psi}) \in K^2 \setminus \{(0,0)\}\}$, where $f_{(\overline{\varphi}, \overline{\psi})} \in K[\gamma_i, \delta_i \mid i = 1, \ldots, p]$ is the valuation of f in $(\varphi, \psi) = (\overline{\varphi}, \overline{\psi})$.

Moreover, we have that $Span_K\{f_{(\overline{\varphi}, \overline{\psi})}, (\overline{\varphi}, \overline{\psi}) \in K^2 \setminus \{(0,0)\}\} = Span_K\{a_0, \ldots, a_p\}$.

Indeed, we have that $Span_K\{f_{(\overline{\varphi}, \overline{\psi})}, (\overline{\varphi}, \overline{\psi}) \in K^2 \setminus \{(0,0)\}\} \subseteq Span_K\{a_0, \ldots, a_p\}$,

so we need to prove the other inclusion. Without losing generality we put $\overline{\psi} = 1$. It's sufficient to show that there exist $\underline{k}' = (k_0', \ldots, k_p') \in K^{p+1}$ such that the following equality holds:

$$\sum_{i=0}^{p} k_i a_i = \sum_{i=0}^{p} k_i' \overline{\varphi}_i^p a_i,$$

where $k_i, \overline{\varphi}_i, k_i' \in K$ and $\overline{\varphi}_i$ are pairwise distinct. It is equivalent to solve the linear system $\underline{k} = A\underline{k}'$, where $\underline{k} = (k_0, \ldots, k_p) \in K^{p+1}$, and A is a square Vandermonde matrix (see Lemma 4.6). Since $\overline{\varphi}_i$ are pairwise distinct, the matrix A is invertible and the system $\underline{k} = A\underline{k}'$ is compatible.

In conclusion, remembering that the functions a_i for $i = 0, \ldots, p$ are linearly independent, if we define $c_i' := (\pi^*)^{-1}(a_i)$ for $i = 0, \ldots, p$, we have that $\{c_0', \ldots, c_p'\}$ is a basis of $SI(Q, p\underline{h})_\partial$ as well as $\{c_i := c_i' f', i = 0, \ldots, p\}$ is a basis of $SI(Q, \underline{d})_\partial$.

In particular, we note that

$$c_0 := (\pi^*)^{-1}(a_0)f' = (\pi^*)^{-1}(f_{(1,0)})f' = c^{V(1,0)},$$

(similarly for c_p and $c_0 + \ldots + c_p$ that are the Schofield semi-invariants corresponding to $V(0,1)$ and $V(1,1)$, respectively). It's known (see [DR] Section 5 and 6) that the modules $V(1,0)$, $V(0,1)$, $V(1,1)$ are consecutive extensions of $E_{i,j}, E'_{r,s}, E''_{t,m}$, respectively, such that $\sum \underline{d}_{E_{i,j}} = \underline{h}$, $\sum \underline{d}_{E'_{r,s}} = \underline{h}$ and $\sum \underline{d}_{E''_{t,m}} = \underline{h}$, respectively. Applying the Proposition 1.1 c), we obtain the relations in (5).

(b) By Step 1 and point *(a)*, we have that the algebra $SI(Q, \underline{d})$ is of the form:

$$\frac{K[c_0, c_1, \ldots, c_p, c^{E_{i,j}}, c^{E'_{r,s}}, c^{E''_{t,m}}]}{I},$$

where I is an ideal which contains the ideal J generated by the relations in (5). To prove that the two ideals J and I are the same ideal, we need to show that the epimorphism μ

$$T := \frac{K[c_0, c_1, \ldots, c_p, c^{E_{i,j}}, c^{E'_{r,s}}, c^{E''_{t,m}}]}{J} \xrightarrow{\mu} SI(Q, \underline{d})$$

is an isomorphism.

We are going to show that the dimensions of weight spaces in both rings are equal. If we look at T and $SI(Q, \underline{d})$ as $\bigoplus_\sigma T_\sigma$ and $\bigoplus_\sigma SI(Q, \underline{d})_\sigma$ respectively, we recognize an epimorphism between the corresponding graded components

$$T_\sigma \longrightarrow SI(Q, \underline{d})_\sigma.$$

Then, we have

$$dim T_\sigma \geq dim SI(Q, \underline{d})_\sigma. \tag{7}$$

Now, using the same arguments of the proof of Proposition 4.5, we have that

$$SI(Q, \underline{d})_{\langle \underline{\alpha}, - \rangle} \cong SI(Q, \underline{d})_{\langle m\underline{h}, - \rangle},$$

where $\underline{\alpha}$ is a regular dimension vector with the canonical decomposition (see (1)) $\underline{\alpha} = m\underline{h} + \underline{\alpha}'$ and $\underline{d}$ has the canonical decomposition $\underline{d} = p\underline{h} + \underline{d}'$.

Then, applying Proposition 4.5 we have that

$$dim \ SI(Q, \underline{d})_{\langle \underline{\alpha}, - \rangle} = \binom{p + m}{m}, \tag{8}$$

and, by (7) and (8), we obtain the inequality:

$$dim \ T_{\langle \underline{\alpha}, - \rangle} \geq dim \ SI(Q, \underline{d})_{\langle \underline{\alpha}, - \rangle} = \binom{p + m}{m}. \tag{9}$$

Thus, if we prove that

$$dim\, T_{\langle \underline{\alpha}, - \rangle} \leq \binom{p+m}{m}, \tag{10}$$

by (9) and (10) follows that μ is an isomorphism.

First of all, we observe that $f \in T_{\langle \underline{\alpha}, - \rangle}$ if and only if f is a polynomial equal to the following one

$$\sum_{(\underline{j}, n_{i,j}, n'_{r,s}, n''_{t,m})} k_{\underline{j}, n_{i,j}, n'_{r,s}, n''_{t,m}} c_0^{j_0} \cdots c_p^{j_p} \prod_{(i,j)} (c^{E_{i,j}})^{n_{i,j}} \prod_{(r,s)} (c^{E'_{r,s}})^{n'_{r,s}} \prod_{(t,m)} (c^{E''_{t,m}})^{n''_{t,m}},$$

where $k_{\underline{j}, n_{i,j}, n'_{r,s}, n''_{t,m}} \in K$ and the sum is over all $(\underline{j}, n_{i,j}, n'_{r,s}, n''_{t,m})$ such that

$$\underline{\alpha} \overset{\star}{=} (\sum_{s=0}^{p} j_s) \underline{h} + \sum_{(i,j)} n_{i,j} \underline{d}_{E_{i,j}} + \sum_{(r,s)} n'_{r,s} \underline{d}_{E'_{r,s}} + \sum_{(t,m)} n''_{t,m} \underline{d}_{E''_{t,m}}.$$

Using the relations (5), we transform the polynomial f into the following one:

$$\sum_{(\underline{i}, m_{i,j}, m'_{r,s}, m''_{t,m})} k_{\underline{i}, m_{i,j}, m'_{r,s}, m''_{t,m}} c_0^{i_0} \cdots c_p^{i_p} \prod_{(i,j)} (c^{E_{i,j}})^{m_{i,j}} \prod_{(r,s)} (c^{E'_{r,s}})^{m'_{r,s}} \prod_{(t,m)} (c^{E''_{t,m}})^{m''_{t,m}},$$

with $i_0 + \ldots + i_p = m'$, that is equivalent to change the decomposition ($\star$) into the following

$$\underline{\alpha} \overset{\star\star}{=} \sum_{l=1}^{m'} \underline{h} + \sum_{(i,j)} m_{i,j} \underline{d}_{E_{i,j}} + \sum_{(r,s)} m'_{r,s} \underline{d}_{E'_{r,s}} + \sum_{(t,m)} m''_{t,m} \underline{d}_{E''_{t,m}}.$$

By the uniqueness of the canonical decomposition, m' is equal to m, the multiplicity of h in the canonical decomposition of $\underline{\alpha}$. Moreover, we know that the only linear relations among the dimension vectors $\underline{h}, \underline{e}_i, \underline{e}'_i, \underline{e}''_i$ in the canonical decomposition (1) are the following: $\underline{h} = \sum_{i \in I} \underline{e}_i = \sum_{i \in I'} \underline{e}'_i = \sum_{i \in I''} \underline{e}''_i$, then we have that the weights of the generators of T are linearly independent except the relations

$$\underline{h} = \sum_{(i,j)} \underline{d}_{E_{i,j}} = \sum_{(r,s)} \underline{d}_{E'_{r,s}} = \sum_{(t,m)} \underline{d}_{E''_{t,m}}.$$

In conclusion, the decomposition ($\star\star$) is unique and f is equal to:

$$\sum_{\underline{i}} k_{\underline{i}} c_0^{i_0} \cdots c_p^{i_p} \Big(\prod_{(i,j)} (c^{E_{i,j}})^{m_{i,j}} \prod_{(r,s)} (c^{E'_{r,s}})^{m'_{r,s}} \prod_{(t,m)} (c^{E''_{t,m}})^{m''_{t,m}} \Big).$$

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Thus, we have that

$$dim\, T_{\langle \alpha, - \rangle} \leq \binom{p+m}{m}, \tag{11}$$

as requested. $\diamond$

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